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ABSTRACT

Mean low-level temperature and wind profiles were constructed for 44 cases of free convection using the data for O'Neill, Nebraska, during July and August 1956. Based upon the expression for the normalized logarithmic wind shear first suggested by Ellison and later refined by Panofsky, a theoretical formula for the Deacon profile number as a function of the Richardson number was derived, and values of the Deacon profile number were computed. One of the parameters entering into this theoretical formula is the ratio of the eddy diffusivities for heat and momentum. This parameter was, in turn, computed from Priestley's expression for the dimensionless heat flux for free-convective cases. In using observed wind data from the mean profile in order to verify the theoretical computations of β, some marked discrepancies occurred above the 100 cm level. These were due to inconsistent wind speed readings, and it was necessary to employ control data based on neutral profiles to correct the wind speeds. When this was done, the theoretical and observed Deacon profile numbers were in very good agreement.

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LIST OF SYMBOLS USED

Symbol	Definition
u	Wind Speed
u_*	Friction Velocity
k	Von Karmann Constant
z	Height above the Ground
β	Deacon profile number
Ri	Richardson Number
g	Acceleration of Gravity
θ	Potential Temperature
α	Monin-Obukhov Constant
L	Monin-Obukhov Scale Length
c_p	Specific Heat of Air
ρ	Density of Air
H	Vertical Flux of Heat by Turbulent Diffusion
K_H	Eddy Diffusivity for Heat Conduction
K_M	Eddy Diffusivity for Heat Momentum
S	Normalized Logarithmic Wind Shear
Rf	Flux Richardson Number
τ	Constant Eddy Stress of the Surface Layer
γ	Ratio of Convective to Mechanical Energy Sources of Turbulent Energy

Symbol	Definition
$z_{1,2}$	Geometric Mean of z_1 and z_2
H_*	Non-Dimensional Heat Flux
γ'	$(K_H/K_M)\gamma$
$\bar{\beta}$	Integral Mean Value of β
RMS	Root Mean Square

1. Introduction.

The Deacon profile number may be defined by

$$\frac{\partial u}{\partial z} = \frac{u_*}{k} z^{-\beta} \qquad (1)$$

where:

- u_* = the friction velocity
- k = the Von Karmann constant and is approximately equal to 0.4
- z = the height above the ground
- β = the Deacon profile number

An alternative definition of β which is equivalent to that in equation (1), provided β and u_* are constant with height in any part of the surface layer, is

$$\beta = -z \frac{\partial^2 u / \partial z^2}{\partial u / \partial z} \qquad (2)$$

With the use of finite differences in the layer 1, 2, 3, the last definition of β implies that an integral mean value is given by

$$\bar{\beta} - 1 = -\frac{\log\left(\frac{u_3 - u_2}{u_2 - u_1}\right)}{\log 2} \qquad (3)$$

In the derivation of equation (3), the successive levels z_1, z_2 and z_3 are taken to be "doubled" levels so that $z_3/z_2 = z_2/z_1 = 2$. Equation (3) affords a means of verifying the value of $\bar{\beta}$ computed by other methods to be discussed later.

The Richardson number is defined by

$$Ri = \frac{g}{\theta} \frac{\partial\theta/\partial z}{\left(\partial u/\partial z\right)^2} \qquad (4)$$

where

g = the acceleration of gravity

θ = the potential temperature within the surface layer

u = the wind speed

The Richardson number represents the ratio of the turbulent energy produced by atmospheric bouyancy to that produced by mechanical friction.

Monin and Obukhov [1954] have introduced a wind profile which has the form

$$\frac{\partial u}{\partial z} = \frac{u_*}{kz}\left(1 + \alpha \frac{z}{L}\right) \qquad (5)$$

in place of the equation $\partial u/\partial z = u_*/kz$ which applies in strictly neutral conditions.

In equation (5), α is a constant estimated by Monin and Obukhov to be 0.6, and L is the Monin-Obukhov scale-length defined according to the pair of equations

$$L = -\frac{u_*^3 \theta c_p \rho}{k g H} \qquad (6)$$

$$H = -K_H \rho c_p \partial\theta/\partial z$$

In (6), c_p and ρ are the specific heat and density of air, respectively, and K_H is the eddy diffusivity for heat conduction. In the surface boundary layer, u_* and H are generally regarded as constant with height, although for H this condition is more appropriate above 1 meter [Priestley, 1959]. It is well known that equation (5) applies only in near neutral conditions. However, Ellison [1957] has recently suggested a more general relationship [see equation (9) below], which covers a wider range of surface-layer stabilities.

In deriving the desired theoretical relationship, the "normalized" logarithmic wind shear S has been used. S is defined by

$$S = \frac{kz}{u_*}\frac{\partial u}{\partial z} \qquad (7)$$

Using S, L and Ri, as defined in equations (7), (6) and (4) respectively, one can derive the equation

$$\frac{z}{L} = -\left(\frac{K_H}{K_M} Ri\right) S = -\left(R_f\right) S \qquad (8)$$

where

K_H/K_M is the ratio of eddy diffusivities for heat and momentum,

$Rf = \frac{K_H}{K_M} Ri$

Rf is the so-called <u>flux Richardson number</u>.

In deriving equation (8) one makes use of the well known definition of the friction velocity

$$u_*^2 = \frac{\tau}{\rho}$$

where $\tau = K_m \rho \, \partial u / \partial z$ is the constant eddy stress of the surface layer, and ρ is the density of air.

On the other hand, Ellison [1957] has designed an interpolation formula for the wind profile which fits observed data under certain limiting conditions of stability. His suggested formula, after some transformations, has the form

$$S^4 - \frac{\gamma z S^3}{L} = 1 \qquad (9)$$

where γ is the ratio of convective to mechanical energy sources of turbulent energy. By utilizing equations (7) and (9), one obtains

$$S = \left(1 - \gamma R_f\right)^{-1/4} = \left(1 - \gamma' R_i\right)^{-1/4}$$

$$\gamma' = \frac{K_H}{K_m} \gamma \qquad (10)$$

For small values of z/L or of Ri, one can easily verify that equation (10) gives the same form upon binomial expansion as that of Monin-Obukhov [equation (5)].

From tests of numerous wind profiles at various micrometeorological sites, Panofsky, Blackadar and McVehil [1960] concluded from equation (10) that $\gamma' = (K_H/K_M)\gamma = 18.0$ gives the best fit. The significance of this value will be seen in Section 4.

2. Computation of the Richardson Number.

A finite difference technique for obtaining Ri at a level $z_{1.2}$ near the midpoint of layer z_1 to z_2 was suggested by Lettau [see pp. 328-329, Lettau and Davidson, 1957], and has the form

$$Ri(z_{1.2}) = -\frac{g\, z_{1.2}(\theta_1 - \theta_2)\ln 2}{\theta_m (u_2 - u_1)^2} \tag{11}$$

where z_1 and z_2 are "successive doubled" levels, and $z_{1.2}$ is the geometric mean level of z_1 and z_2 defined by $z_{1.2} = \sqrt{z_1 z_2} = \sqrt{2}\, z_1$. In equation (11), θ_2 and u_2 apply to the top of the layer, and θ_1 and u_1 apply to the lower boundary, while Ri is considered applicable at the geometric mean height $z_{1.2}$. Note that Ri is negative in unstable conditions.

3. The treatment of the data.

The data employed were selected from Table 8.1 and 8.2 of the record of Project Prairie Grass [Barad et al, 1958] and were restricted to cases of free convection. Priestley's criterion for free convection, $-Ri_{1.5} > 0.03$, has been used, where the subscript 1.5 refers to a measurement of Ri at 1.5 meters. From the wind data of Project Prairie Grass, 44 cases of free convection were selected, and from these a composite wind profile was computed. The mean wind speeds at each doubled level are listed in Table 1. For these same free-convective cases,

[illegible] [illegible] [illegible] temperature profile, the corresponding Ri values for the various "centered" coupled levels ($z_{i,j}$) have been computed.

In Table I, numbers with an asterisk have been obtained by extrapolation using the logarithm of height as the independent variable. For the wind speed, simple linear extrapolation has been employed; this is equivalent to assuming that a unique logarithmic wind profile exists in layers below 50 cm. Based upon the work of Davidson and Barad [1956], who showed that β approaches unity near $z = 0$, this is a rather well accepted approximation [cf., also the work of Panofsky et al, 1960]. The temperature at the level 0.25 cm was obtained by a polynomial extrapolation technique, using equal logarithmic intervals of height as an independent variable.

Note that potential temperature and wind speed differences have been entered in Table I directly beneath the layer geometric mean, which appears in row 4.

4. Computation of the ratio of the eddy diffusivities for heat and momentum.

Priestley has shown by dimensional analysis that the non-dimensional heat flux H_* has the form

$$H_* = \frac{H}{\rho c_p \left(g/\theta\right)^{1/2} \left|\partial\theta/\partial z\right|^{3/2} z^2}$$

TABLE 1. Mean micrometeorological data obtained from 44 free convective cases at successive levels of the Texas A. and M. installation at O'Neill, Nebraska during the months of July and August 1956.

Level (cm)	6.25	12.5	25.0	50.0	100.0	200.0	400.0	800.0	1600.0
Potential Temp. (°K)	307.46*	306.06	305.26	304.29	303.48	302.79	302.23	301.76	301.38
Windspeed ($cm\ sec^{-1}$)	194.8*	253.3*	311.8	370.4	421.7	464.7	512.8	548.4	585.8

Layer geometric mean (cm)	8.84	17.68	35.35	70.70	141.42	282.84	565.68	1131.36
$\theta_1 - \theta_2$ (°K)	1.42*	0.80	0.97	0.81	0.68	0.57	0.47	0.38
$u_2 - u_1$ ($cm\ sec^{-1}$)	58.5*	58.3*	58.5	51.3	43.0	48.1	35.6	37.4
$-Ri \times 10^{-3}$	8.3*	9.2*	24.4	48.7	116.7	156.6	471.5	692.8

* Numbers with an asterisk are extrapolated values.

and in cases of free convection, H_* has the average value 0.[illegible] However, H_* may be expressed in the form

$$H_* = \frac{K_H}{K_M} k^2 \left(\frac{K_H}{K_M} \gamma - \frac{1}{Ri} \right)^{1/2} \tag{12}$$

by applying equations (4) and (10) with the first form of H_*. Thus by equation (12) with $H_* = 0$, the $(Ri, K_H/K_M)$ relationship for cases of free convection is

$$\frac{K_H}{K_M} \left(\frac{K_H}{K_M} \gamma - \frac{1}{Ri} \right)^{1/2} = 5.625, \tag{13}$$

using the value $k = 0.4$ for the von Karman constant. In using equation (13), $\gamma' = (K_H/K_M)\gamma = 18.0$ was assumed to hold exactly at the level 1.41 meters, that is, where $Ri = -0.1167$. This gives the following results at $z = 1.41$ meters:

$$\frac{K_H}{K_M} = 1.091 \quad \text{and} \quad \gamma = 16.49$$

Henceforth, $\gamma = 16.49$ was treated as a constant at all other layer centers of Table 1 and the $(Ri, K_H/K_M)$ relationship becomes

$$\left(\frac{K_H}{K_M} \right)^2 \left[\frac{K_H}{K_M} - \frac{1}{Ri} \right] = 1.919 \tag{14}$$

The ratio K_H/K_M is thus obtained as the solution of the cubic equation (14) at all other geometric mean levels. Table 2 shows the values of K_H/K_M computed by equation (14)

at the various doubled levels using the computed Richardson number of Table 1.

Since Panofsky et al [1960] obtained good agreement, on the average, using $\gamma' = (K_H/K_M)\,\gamma = 18.0$, this value of γ' was assumed to hold exactly in this study at the level 1.41 meters. The main justification of this assumption is that this height is close to the geometric mean of the height range of the wind levels under investigation. It turns out that the choice $\gamma'' = 18.0$ at 1.41 meters, results in values of γ' slightly greater than 18.0 above 1.41 meters, but somewhat smaller below this level. Moreover, when these γ' values have been plotted versus log z for 25 cm $\leq z \leq$ 1600 cm, an average value of γ' very close to 18.0 results.

TABLE 2. Computed values of $\frac{K_H}{K_M}$ at layer geometric mean levels, using equation (14) for free-convective cases at O'Neill, Nebraska, during July and August 1956.

Level (cm)	6.25		12.5		25.0		50.0		100.0		200.0		400.0		800.0		1600.0
Layer geometric mean (cm)		8.84		17.68		35.35		70.70		141.42		282.84		565.68		1131.36	
$\frac{K_H}{K_M}$		0.486*		0.519*		0.745		0.938		1.091		1.126		1.201		1.214	

* Numbers with an asterisk are extrapolated values, based on extrapolations of Table 1.

5. The β - Ri relationship.

From the definition of $S = \frac{kz}{u_*} \partial u/\partial z$;

logarithmic differentiation with respect to z leads to

$$\frac{1}{S}\frac{\partial S}{\partial z} = \frac{1}{z}\left(1 + z\frac{\partial^2 u/\partial z^2}{\partial u/\partial z} - \frac{z}{u_*}\frac{\partial u_*}{\partial z}\right) \qquad (15)$$

From equation (2) the second term within the parentheses is recognized to be $-\beta$. Also upon applying logarithmic differentiation to the normalized logarithmic wind shear $S = (1 - \gamma\, Ri)^{-1/4}$ one obtains

$$\frac{1}{S}\frac{\partial S}{\partial z} = -\frac{1}{4}\,\frac{\partial/\partial z\,(1 - \gamma' Ri)}{(1 - \gamma' Ri)} \qquad (16)$$

Elimination of $\frac{1}{S}\partial S/\partial z$ from equations (15) and (16) leads to

$$\beta - 1 = \left[\frac{1}{4}\, z\,\frac{\partial/\partial z\left(1 - \gamma \frac{K_H}{K_M} Ri\right)}{\left(1 - \gamma \frac{K_H}{K_M} Ri\right)}\right] \qquad (17)$$

assuming u_* to be constant in the layer.

Hence integration of equation (17) from z_1 to z_2 yields the mean value $\bar{\beta}$ for the layer given by

$$\bar{\beta} - 1 = \frac{1}{4}\left[\ln\left\{1 - \gamma\left(\frac{K_H}{K_M}\right)Ri_2\right\} - \ln\left\{1 - \gamma\left(\frac{K_H}{K_M}\right)Ri_1\right\}\right] \Big/ \ln\frac{z_2}{z_1} \qquad (18)$$

where K_H/K_M is also a function of Ri. This equation is then the <u>theoretical</u> β – Ri relationship. The value $\bar{\beta}$ is to be associated with the geometric mean height

$z_{1,2} = \sqrt{z_1 z_2}$ in the layer z_1 to z_2. Application of equation (18) is made for layers between successive doubled levels of Table 2 at which a value of K_H/K_M is known.

Since $\gamma = 16.49$ is treated as a constant within the surface layer, equation (18) becomes

$$\bar{\beta} - 1 = \frac{\log_e\left[1 - 16.49\left(\frac{K_H}{K_M}\right)_{12} Ri_2\right] - \log_e\left[1 - 16.49\left(\frac{K_H}{K_M}\right)_{12} Ri_1\right]}{4 \log_e 2} \tag{19}$$

The results obtained using this equation, together with the data of Table 1 and 2, are listed in Table 3. In this table, the following three different types of $\bar{\beta}$ values are tabulated against elevation:

$\bar{\beta}_{I}$, the value resulting from equation (19) with K_H/K_M values taken from Table 2.

$\bar{\beta}_{II}$, the value resulting from equation (18) with $(K_H/K_M)\,\gamma = 18.0$ at all levels, so that satisfies

$$\bar{\beta}_{II} - 1 = \frac{\log_e(1 - 18.0\, Ri_2) - \log_e(1 - 18.0\, Ri_1)}{4 \log_e 2} \tag{20}$$

$\bar{\beta}_{III}$, the value resulting from equation (3).

It should be noted that $\bar{\beta}_{I}$ and $\bar{\beta}_{II}$ are in relatively close agreement, but some unusual discrepancies occur in $\bar{\beta}_{III}$. This difficulty is dealt with at more length in Section 6.

6. Correction for $\bar{\beta}_{III}$ values.

Some inconsistency occurs in computing $\bar{\beta}$ using direct wind data. At the 200 cm level, the $\bar{\beta}_{III}$ value was very low in comparison with the surrounding $\bar{\beta}$'s. This error occurred because the vertical wind speed increment between the 200 cm and 400 cm level was not consistent with those in adjacent layers. This suggests that there may have been an instrumental error at one or more of these levels. Hence a control data table was made up by selecting data occurring at the time of a neutral wind profile, that is, when the potential temperature is isothermal with height. This usually occurs near sunrise and sunset. Fourteen (14) cases have been selected that meet those requirements and from these a near-neutral wind profile was computed, as shown in Table 4.

A perfect neutral wind profile would have the characteristic of constant $(u_2 - u_1)$ increments between successive doubled levels. However, the values of $(u_2 - u_1)$ in Table 4 indicates that this was not the case. The lowest three layers in Table 4 have $(u_2 - u_1)$ increments which are quite close to their overall mean of 49.6 cm sec^{-1}. This last value indicates the slope of the logarithmic profile which exists under neutral conditions. The remaining three wind increments of Table 4 then indicate percentagewise how much the next three layers deviated from consistency with the lower

TABLE 3. Theoretical and verifying values of $\bar{\beta}$

Level (cm)	6.25		12.5		25.0		50.0		100.0		200.0		400.0		800.0		1600.0
Layer geometric mean (cm)		8.84		17.68		35.35		70.70		141.42		282.84		565.68		1131.66	
$\bar{\beta}_{I}$ by (19)	-		-		1.060		1.115		1.208		1.085		1.349		1.131		
$\bar{\beta}_{II}$ by (20)	-		-		1.035		1.105		1.181		1.075		1.328		1.126		
$\bar{\beta}_{III}$ by (3)	-		-		-		1.189		1.225		0.838		1.433		0.930		

layers. In the last row of Table 4, a correction factor has been included for each layer. For example, the correction factor 0.822 is based on the fact that the mean wind increment recorded is 60.3 cm sec^{-1} rather than 49.6 cm sec^{-1}. In arriving at the corrected wind-speed differences of Table 5, one must enter with the data of Tables 1 and 4. Thus for example, the mean wind difference in the layer 200 to 400 cm is indicated as 48.1 cm sec^{-1} in the non-neutral case of Table 1. However, the data of Table 4 indicates that this reading is overestimated in the neutral case by the factor 1/0.822. Hence, the corrected wind speed increment applicable to the same layer is

$$0.822 \times 48.1 \text{ cm sec}^{-1} = 39.57 \text{ cm sec}^{-1}$$

Similar corrections may then be applied to the other layers.

With the corrected wind differences obtained by this method and displayed in Table 5, corrected $\overline{\beta}_{III}$ values are then obtained from equation (3) and are listed in the last row of the table.

TABLE 4. Control-data table, showing the mean neutral wind profile computed from 14 cases near sunrise and sunset, and correction factors.

Level (cm)	6.25	12.5	25.0	50.0	100.0	200.0	400.0	800.0	1600.0
Mean Wind Speed cm sec^{-1}	-	-	262.5	317.1	365.2	411.5	471.8	523.9	590.[illegible]
$u_2 - u_1$ cm Sec^{-1}		-	-	49.8	49.5	49.0	60.3	52.5	66.7
Correction factors		-	-	1	1	1	0.822	0.948	0.744

TABLE 5. Corrected wind-speed differences using correction factors from Table 4, and corrected β-values.

Level (cm)	6.25	12.5	25.0	50.0	100.0	200.0	400.0	800.0	1600.0
$u_2 - u_1$ cm sec^{-1}	58.5	58.5	58.5	51.3	43.	39.6	33.8	27.9	
$\bar{\beta}_I$ by (19)	-		1.060	1.115	1.206	1.085	1.349	1.131	
$\bar{\beta}_{II}$ by (20	-	1.015	1.035	1.105	1.181	1.075	1.328	1.126	
Corrected $\bar{\beta}_{III}$ by (3)	-	-	-	1.189	1.225	[1.120]	[1.228]	[1.281]	

[ ] indicates corrected values.

Finally as a conclusion, comparison of the values obtained theoretically with those obtained directly from the corrected wind data indicates that the theoretical formulas are well substantiated.

The average difference between $\bar{\beta}_{I}$ and $\bar{\beta}_{III}$ is equal to -0.031 with an RMS error of 0.094. Similarly, the average difference between $\bar{\beta}_{II}$ and $\bar{\beta}_{III}$ is equal to -0.046 with an RMS error of 0.095.

Of the two theoretical formulas, there seems to be a slight advantage to the use of equation (20) over equation (19), except possible below 25 cm, where due to lack of data, comparison is not possible.

BIBLIOGRAPHY

1. Barad, M. L., 1958: Project Prairie Grass, a field program in diffusion, Vol. II, Geophys. Res. Pap., No. 59, 209 pp.

2. Blackadar, A. K., H. A. Panofsky, G. E. McVehil, and S. H. Johnston, 1960: Structure of turbulence and mean wind profiles within the atmospheric boundary layer. Penn State Univ., University Park, Pa. (under contract AF 19 (604)-5251 to Air Force Cambridge Laboratories), 81 pp.

3. Davidson, B., and M. L. Barad, 1956: Some comments on the Deacon wind profile, Trans. Am. Geophys. Union.

4. Ellison, T. H., 1957: Turbulent transport of heat and momentum from an infinite rough plane. J. Fluid Mech., Vol. 2, pp 456-466.

5. Lettau, H., and B. Davidson, ed., 1957: Exploring the atmosphere's first mile, vols. 1 and 2, Pergamon Press, London; 578 + xiv pp.

6. Martin, F.L., unpublished manuscript, 1961: A relationship between the Deacon profile number and the Richardson number under convective conditions.

7. Monin, A. S., and A. M. Obukhov, 1954: Osnovnye zakonomernosti turbulentnogo peremoshivania v prizemnon sloe atmosfery. (Basic law of turbulent mixing in the ground layer of the atmosphere). Akademia Nauk SSSR Leningrad, Geofizicheskii Institut, Trudy No. 24 (151), pp. 163-187.

8. Panofsky, H. A., A. K. Blackadar and G. E. McVehil, 1960: The diabatic wind profile. Q. J. Roy. Meteor. Soc., Vol. 86, pp. 390-398.

9. Priestley, C. H. B., 1959: Turbulent transfer in the lower atmosphere. Univ. Chicago Press; 130 + vii pp.